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Fractional trajectories: Decorrelation versus friction

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HIGHLIGHTS

- We study the connection between fractional calculus and subordination processes.
- A fractional trajectory is equivalent to the ensemble average trajectory resulting from subordination.
- The internal friction effects associated to fractional derivatives can alternatively be explained by decorrelation.
- We use subordination theory to study the relaxation toward the equilibrium state induced by the fractional derivatives.
- As an example we apply the theory to the fractional Lotka–Volterra ecological model.

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1. Introduction

In the last few years there has been an increasing interest in the adoption of fractional calculus in time to study a wide set of issues, ranging from economics [1–3] to ecology [4,5] to allometry [6] to bioengineering [7]. Fractional calculus more readily caught the attention of engineers since it allows for the simulation of and control over a more diverse range of complex system behaviors [8]. The primary motivation for the use of time fractional derivatives has been that they generate solution trajectories with memory, and therefore give a more realistic representation to many complex processes found in nature. This memory effect has produced what is usually referred to as an intrinsic damping in the system [9–11], pointed out by many authors to be equivalent to assuming that the process under study is in contact with a dissipative environment. A fundamental example of this so-called intrinsic friction effect happens with the fractional simple harmonic oscillator, where replacing the ordinary time derivatives with fractional derivatives leads to damped oscillations, even without considering an

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external friction force [10,11]. Perhaps this interpretation of the fractional harmonic oscillator stems from the success of the fractional calculus description of viscoelastic material properties, where memory and internal friction are plausible [12–15].

The goal of the present article is to suggest the importance of another interpretation, that of decorrelation. We show that a single fractional trajectory can be represented as the average over an infinite number of ordinary differential trajectories, each trajectory keeping an internal time using stochastic clocks that operate independently of one another [16]. When this is the case the dissipation effects noticed by the majority of investigators in this field are shown herein to actually be a form of decorrelation. To support our theoretical arguments we provide an ecological example in an application of the theory to the Lotka–Volterra model. It has to be stressed, however, that the new perspective is not limited to ecology, and it is expected to transform the interpretations currently adopted in many other fields of research, as discussed in the concluding section.

Historically, the correct physical interpretation of fractional calculus has been elusive. Perhaps the most well known example of a dynamical process leading to a fractional calculus description lies in the anomalous diffusion processes governed by the continuous time random walk (CTRW) theory [17–19]. From a generalized master equation approach, it has been shown that an ensemble of particles undergoing a CTRW in an external field can be represented by a time-fractional Fokker–Planck equation [20]. The CTRW is a specific example of a much more general theory, usually referred to as subordination [21–23] because, in a certain sense, subordination is a diffusion process along a prespecified path. We emphasize that the theory presented here is general and goes beyond diffusion to include any trajectory resulting from the integration of fractional differential equations in time.

In Section 2 we provide a general demonstration of the new perspective on fractional calculus using an operator procedure and the Mittag-Leffler function. Section 3 is an application of the theory to the Lotka–Volterra system, including a comparison between the fractional Lotka–Volterra trajectory and the average trajectory resulting from the subordination process. In addition we discuss the time required to relax to the equilibrium state induced by the adoption of fractional calculus. Finally Section 4 is devoted to discussing the conceptual consequences that the new fractional derivative perspective may have in other fields of investigation.

2. General method

In this section we demonstrate the equivalence between a fractional trajectory that is the solution of a Caputo fractional differential equation, and the ensemble average trajectory that results from a subordination process. We consider only fractional derivatives of the Caputo type, noting that a Riemann–Liouville approach would be equivalent as long as the initial conditions are properly specified.

2.1. Fractional trajectory

First let us consider the fractional differential equation

$$\frac{d^\alpha}{dt^\alpha}\mathbf{V}(t) = O\mathbf{V}(t), \tag{1}$$

where $0 < \alpha < 1$ and O is an operator, either linear or nonlinear, acting on the vector $\mathbf{V}(t)$. With this general notation we can describe any dynamical system of interest. To solve Eq. (1) we use the Laplace transform of a Caputo fractional derivative [24],

$$\pounds\left\{\frac{d^\alpha}{dt^\alpha}\mathbf{V}(t); s\right\} = s^\alpha\widehat{\mathbf{V}}(s) - s^{\alpha-1}\mathbf{V}(0), \tag{2}$$

to find that

$$\widehat{\mathbf{V}}(s) = \frac{1}{s - Os^{1-\alpha}}\mathbf{V}(0). \tag{3}$$

This is the Laplace transform of a Mittag-Leffler function, and moving back to the time domain yields the fractional trajectory

$$\mathbf{V}(t) = E_\alpha(Ot^\alpha)\mathbf{V}(0). \tag{4}$$

Note that the Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k + 1)}, \tag{5}$$

where Γ denotes the gamma function [19]. With this series representation it can be seen that there is no ambiguity in the action of the operator in the argument of the Mittag-Leffler function, since it is similar to the exponential solution of an ordinary derivative in that it requires the calculation of a series of operator moments of the form $O^n\mathbf{V}(0)$.

Recall the main idea is that the fractional trajectory described by Eq. (4), which is the solution to the fractional differential equation of Eq. (1), is equivalent to a subordination process. To establish this equivalence we will now move to the process of subordination.

2.2. Subordination process

Subordination implies the existence of two different notions of time [23]. One is the operational time τ, which is the internal time of a single unit, with a unit generating the ordinary dynamics of a non-fractional system. The other notion is experimental time t; the time as measured by the clock of an external observer. Typically in the operational time frame the temporal behavior of a unit is regular and evolves exactly according to the ticks of a clock. Therefore it is assumed that the system's trajectory in operational time is well defined and given by $\mathbf{V}(\tau)$, which is the solution to the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{V} = O\mathbf{V}. \tag{6}$$

The discrete solution to Eq. (6) can be found recursively through

$$\mathbf{V}(\tau + \Delta\tau) = (1 + O\Delta\tau)\mathbf{V}(\tau), \tag{7}$$

provided that the integration time step is sufficiently small compared to the eigenvalues of the operator O, a condition which we informally state as

$$O\Delta\tau \ll 1. \tag{8}$$

Moving from the initial condition of the system and replacing the continuous operational time τ with its discrete counterpart n we can write

$$\mathbf{V}(n) = (1 + O\Delta\tau)^n\mathbf{V}(0), \tag{9}$$

with the concise notation

$$\mathbf{V}(n) \equiv \mathbf{V}(n\Delta\tau). \tag{10}$$

In operational time the units' behavior appears ordinary, but to an experimenter observing the units their temporal behavior appears erratic, evolving in time then abruptly freezing in different states for extended time periods. Because of the random nature of the experimental time evolution of the units, the subordination process involves an ensemble average over many units each evolving according to its own internal clock, independent of one another. Making an ensemble average over a large number of units results in a smooth average trajectory, which is equivalent to the fractional trajectory.

To find the average behavior we can move from the operational time solution to the experimental time solution using the probability of the occurrence of an event

$$\mathbf{V}(t) = \sum_{n=0}^{\infty}\mathbf{V}(n)\int_0^t \mathrm{d}t'\psi_n(t')\Psi(t - t'). \tag{11}$$

Actually here we should write $\mathbf{V}(t)$ as $\langle\mathbf{V}(t)\rangle$, but for convenience we drop the brackets, it being evident that the trajectory resulting from the subordination process inherently involves an ensemble average.

To interpret the physical meaning of Eq. (11), consider each tick of the internal clock n of a unit measured in experimental time as an event. Since the observation is made in experimental time, the time intervals between events are random. We assume that the waiting times between consecutive events are identically distributed independent random variables. The integral in Eq. (11) is then built up according to renewal theory [25]. After the nth event the unit changes from state $\mathbf{V}(n)$ to $\mathbf{V}(n + 1)$, where it remains until the action of the next event. The sum over n takes into account the possibility that any number of events could have occurred prior to an observation at experimental time t. The events occur probabilistically with a waiting-time probability distribution function (*pdf*) $\psi(t)$ and survival probability $\Psi(t)$. The waiting-time *pdf* is related to the survival probability through

$$\psi(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t). \tag{12}$$

Taking advantage of the renewal nature of the events, the waiting-time *pdf* for the nth event in a sequence is connected to the previous event by

$$\psi_n(t) = \int_0^t \mathrm{d}t'\psi_{n-1}(t')\psi(t - t'). \tag{13}$$

To find an analytical expression for the behavior in experimental time it is convenient to study the Laplace transform of Eq. (11),

$$\widehat{\mathbf{V}}(s) = \widehat{\Psi}(s)\sum_{n=0}^{\infty}\mathbf{V}(n)\left[\widehat{\psi}(s)\right]^n, \tag{14}$$

where the relation $\widehat{\psi}_n(s) = \left[\widehat{\psi}(s)\right]^n$ was used according to Eq. (13). With the discrete time solution in operational time, Eq. (9), this can be written as

$$\widehat{\mathbf{V}}(s) = \widehat{\Psi}(s)\sum_{n=0}^{\infty}(1+O\Delta\tau)^n\mathbf{V}(0)\left[\widehat{\psi}(s)\right]^n. \tag{15}$$

Performing the sum and noting the relationship given by Eq. (12) we find

$$\widehat{\mathbf{V}}(s) = \frac{1-\widehat{\psi}(s)}{s}\frac{1}{1-(1+O\Delta\tau)\widehat{\psi}(s)}\mathbf{V}(0). \tag{16}$$

This can be expressed in the typical form

$$\widehat{\mathbf{V}}(s) = \frac{1}{s-O\Delta\tau\widehat{\Phi}(s)}\mathbf{V}(0), \tag{17}$$

where

$$\widehat{\Phi}(s) = \frac{s\widehat{\psi}(s)}{1-\widehat{\psi}(s)} \tag{18}$$

is the Montroll–Weiss memory kernel [17].

2.3. Establishing an equivalence between the subordination process and the fractional trajectory

The subordination process can be connected to the fractional trajectory when the survival probability for the events is given by the Mittag-Leffler function

$$\Psi(t) = E_\alpha(-At^\alpha), \tag{19}$$

where the asymptotic power-law region of the Mittag-Leffler function scales with the same index α as the fractional derivative in Eq. (1). In order for the Mittag-Leffler function to be compatible with a survival probability the index must be in the interval $0 < \alpha < 1$, which sets the interesting condition that the mean waiting time between consecutive events is infinite since

$$\langle t\rangle \equiv \int_0^\infty \tau\psi(\tau)\,\mathrm{d}\tau = \int_0^\infty E_\alpha(-A\tau^\alpha)\mathrm{d}\tau \tag{20}$$

diverges due to the fact that asymptotically the Mittag-Leffler function is an inverse power-law. This inverse power-law or scale-free condition corresponds to nonstationary processes for $0 < \alpha < 1$ and leads to a form of ergodicity breakdown [26]. Ergodicity breakdown has interesting statistical consequences at the level of single trajectories [27,28] and has been observed experimentally in anomalous diffusion processes in living cells [29,30].

With the choice of survival probability given by the Mittag-Leffler function the Montroll–Weiss memory kernel is exactly

$$\widehat{\Phi}(s) = As^{1-\alpha}, \tag{21}$$

and Eq. (17) reduces to

$$\widehat{\mathbf{V}}(s) = \frac{1}{s-O\Delta\tau As^{1-\alpha}}\mathbf{V}(0). \tag{22}$$

Consequently, under the condition

$$\Delta\tau A = 1. \tag{23}$$

Eq. (22) becomes identical to the Laplace transform of the fractional derivative and the solution is again the Mittag-Leffler function given in Eq. (4). The result is that a fractional trajectory can be represented as the ensemble average trajectory of a subordination process. This result is exact when the survival probability for events is a Mittag-Leffler function and the conditions of Eqs. (8) and (23) are satisfied. The requirement of a survival probability that is a Mittag-Leffler function can be relaxed to generic survival probabilities that share the asymptotical inverse power-law behavior of the Mittag-Leffler function, by way of a stochastic central limit theorem [31].

2.4. Eigenfunction expansion

Let us examine the condition defined by Eq. (8) more carefully. The generic system described by Eq. (6) involves a spectrum of eigenvalues γ_j of the operator O. The eigenfunctions associated with this problem are given by

$$\phi_j(\mathbf{V}) \equiv \left(\mathbf{V}, \chi_j\right) \tag{24}$$

where $\boldsymbol{\chi}_j$ are the eigenfunctions of the adjoint operator $O^\dagger$

$$O^\dagger \boldsymbol{\chi}_j = -\gamma_j \boldsymbol{\chi}_j \tag{25}$$

normalized such that the inner product

$$\left(\mathbf{v}_j, \boldsymbol{\chi}_k\right) = \delta_{j,k} \tag{26}$$

where $\mathbf{v}_j$ is an eigenvector of O. The dynamics of the eigenvectors in operational time are given by

$$\frac{\mathrm{d}\phi_j}{\mathrm{d}\tau} = \left(\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}\tau}, \boldsymbol{\chi}_j\right) = (O\mathbf{V}, \boldsymbol{\chi}_j) = (\mathbf{V}, O^\dagger \boldsymbol{\chi}_j) = -\gamma_j \left(\mathbf{V}, \boldsymbol{\chi}_j\right) = -\gamma_j \phi_j, \tag{27}$$

which has the exponential solution

$$\phi_j(\mathbf{V}, \tau) = e^{-\gamma_j \tau} \phi_j(\mathbf{V}_0). \tag{28}$$

Consequently, the state vector has the eigenfunction decomposition, as long as the operator O has a complete spectrum,

$$\mathbf{V}(\tau) = \sum_j \left(\mathbf{V}, \boldsymbol{\chi}_j\right) \mathbf{v}_j = \sum_j \mathbf{v}_j \phi_j \left(\mathbf{V}, \tau\right). \tag{29}$$

We can select a discrete set of points along the continuous curve defined by the eigenfunctions separated by a time distance $\Delta\tau$ to obtain

$$\phi_j(\mathbf{V}, n\Delta\tau) = e^{-\gamma_j n \Delta\tau} \phi_j(\mathbf{V}_0), \tag{30}$$

where we have replaced the continuous subordination time τ with the discrete subordination time n. The subordination of this single eigenfunction following the previous argument results in the exact expression

$$\widehat{\phi}_j(\mathbf{V}, s) = \frac{1}{s + (1 - e^{-\gamma_j \Delta\tau})\widehat{\Phi}(s)} \phi_j(\mathbf{V}_0). \tag{31}$$

Only when the time step is sufficiently small that the approximation

$$e^{-\gamma_j \Delta\tau} \approx 1 - \gamma_j \Delta\tau \tag{32}$$

is valid over the entire spectrum of eigenvalues of the operator O can we write Eq. (22). When this condition is satisfied we recover the Laplace transform of a Mittag-Leffler function that shares the properties generated by the fractional trajectory in Eq. (3).

This digression provides the interpretation of Eq. (8). If the discrete time step is properly selected then the fractional trajectory will be exactly equivalent to the average trajectory resulting from the subordination process. For nonlinear systems there are infinitely many eigenvalues to consider when choosing the time step $\Delta\tau$, making the study of fractional trajectories as compared to subordination a non-trivial problem that is discussed in detail in the following section.

3. Ecological application

As an ecological application of the theory presented in Section 2, we consider the Lotka–Volterra equations that were originally introduced to describe the population dynamics of a predator–prey system. The fractional system we propose strictly serves a tutorial purpose, and has not been connected to any real ecological observations that we are aware of. To construct a more robust fractional system with a stronger connection to real ecological processes, it is necessary to take into account a proper fractionalization of a two-compartment system, see for example [32,33].

This section is divided into two parts. In the former we analyze the subordinated Lotka–Volterra system and compare it to the Lotka–Volterra fractional trajectory. In the latter part we study the evolution of the fractional and subordinated trajectories toward equilibrium.

3.1. *A comparison between the fractional trajectory and subordination*

The fractional Lotka–Volterra system is described by [4,5]

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} x = ax - bxy \tag{33}$$

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} y = -cy + \delta xy, \tag{34}$$

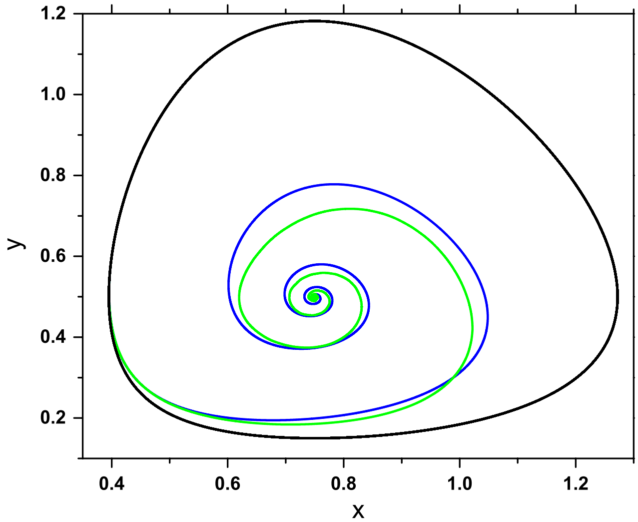

Fig. 1. A comparison of the different Lotka–Volterra trajectories in the predator–prey phase space. The ordinary Lotka–Volterra trajectory in operational time (black) is a cycle. The fractional (blue) and average subordinated (green) Lotka–Volterra trajectories spiral in to the fixed point ($x = 0.75, y = 0.5$). An ensemble of $N = 10^4$ subordinated trajectories was generated to find the smooth average trajectory shown. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

where x represents a prey population and y the predator population. According to the general arguments of the previous section we can define the operator in Eq. (1) as

$$O = (ax - bxy)\frac{\partial}{\partial x} - (cy - \delta xy)\frac{\partial}{\partial y}, \tag{35}$$

with the vector $\mathbf{V}$ having the components x and y. The predictor–corrector integration method is adopted to solve this system of fractional differential equations to generate a fractional trajectory when the fractional index is $\alpha = 0.9$ [34]. For consistency in the numerical simulations we set the coefficients to the values $a = 0.001$, $b = 0.002$, $c = 0.003$, $\delta = 0.004$, and use the initial conditions ($x_0 = 0.4, y_0 = 0.6$) in each case. This fractional trajectory is then compared to the ensemble average trajectory found using the following subordination process.

In operational time the ordinary Lotka–Volterra model is described by the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}x = ax - bxy \tag{36}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}y = -cy + \delta xy. \tag{37}$$

Using the predictor–corrector method with $\alpha = 1$ we numerically integrate the system of differential equations to find the operational time trajectory $\mathbf{V}(n) = (x(n), y(n))$, where we have set the operational time step to be $\Delta\tau = 1$. This ordinary Lotka–Volterra trajectory is shown in Fig. 1. We then go through the subordination process, moving from operational time to experimental time by generating a sequence of random time intervals $\{T_1, T_2, T_3, \ldots\}$, from a waiting-time *pdf* corresponding to a Mittag-Leffler function survival probability [35]. The parameters of the Mittag-Leffler function were set to $\alpha = 0.9$ to match the fractional derivative, and $A = 1$ to fit the condition given by Eq. (23). The subordinated trajectory begins with the initial condition $\mathbf{V}(0) = (x_0, y_0)$. It (the trajectory) stays in this state until an event occurs, at experimental time $t = T_1$ corresponding to the first random waiting time drawn, where it jumps to the next state $\mathbf{V}(1)$. Again it remains in this state until the next event occurs after waiting a time T_2, corresponding to an overall experimental time of $t = T_1 + T_2$. In this way a single subordinated trajectory is constructed, see Fig. 2 for example. Applying the same procedure, an ensemble of subordinated trajectories is created and from it the average over subordinated trajectories is found.

A comparison between the ensemble average trajectory of the subordination process and the fractional trajectory is shown in Fig. 1. Both the fractional trajectory and the average trajectory resulting from the subordination process spiral in to the same equilibrium point, however the two trajectories are not exactly equivalent. Here we are forced to rest on the numerical results of the fractional derivative and subordination algorithms, either or both of which may not accurately represent the unknown theoretical trajectory. If we make the unverified assumption that the fractional derivative algorithm produces an accurate representation of the fractional Lotka–Volterra system, then a possible source of inaccuracy in the subordination algorithm could be the violation of the condition stated in Eq. (8). Due to the nonlinear nature of the Lotka–Volterra system, some of the eigenvalues of the operator O might exceed the inverse of the operational time step. The large eigenvalues govern the short time behavior of the fractional and subordinated Lotka–Volterra trajectories, and the

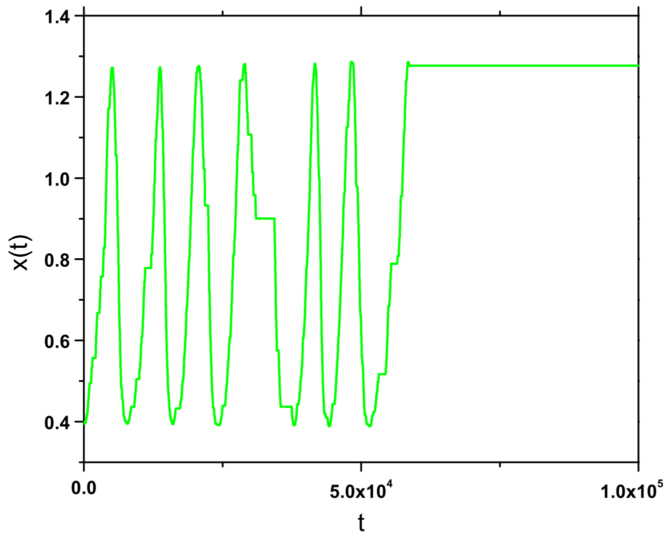

Fig. 2. A single subordinated Lotka–Volterra prey trajectory observed in experimental time is stochastic, erratically freezing in certain population levels for times that can be on the same order as the overall observation time.

difference between the two trajectories may be due to an inaccurate representation in this short time region. In fact, both the fractional trajectory and the average trajectory resulting from the subordination process reach the same equilibrium point, an indication that the smallest eigenvalues of the operator O, determining the asymptotic time behavior, fit the condition of Eq. (8). For nonlinear systems it is an open numerical challenge to find an exact equivalence between the fractional trajectory and subordination at both short and asymptotic times, whereas for linear systems an exact equivalence between the two trajectories can be found for all times.

3.2. Decorrelation and equilibrium

The non-zero fixed point for the ordinary Lotka–Volterra system located at the point

$$(x_{eq}, y_{eq}) = \left(\frac{c}{\delta}, \frac{a}{b}\right) \tag{38}$$

can be found by setting the velocities to zero in Eqs. (36) and (37). This fixed point remains unchanged when moving to the fractional Lotka–Volterra system. An ordinary Lotka–Volterra trajectory is periodic, making one cycle after a period T, and the fixed point can alternatively be calculated by a time average of each population over one cycle [36]. To demonstrate this calculation for x_{eq} we go back to Eq. (37) and rewrite it as

$$\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}\tau}y = -c + \delta x \tag{39}$$

which has the solution

$$ln\left(\frac{y(\tau)}{y_0}\right) = -c\tau + \delta\int_0^\tau \mathrm{d}t\, x(t). \tag{40}$$

After the time period T the trajectory returns to its initial condition, $y(T) = y_0$, so that the logarithm on the left-hand side of Eq. (40) vanishes identically to yield

$$x_{eq} = \frac{c}{\delta} = \frac{1}{T}\int_0^T \mathrm{d}t\, x(t). \tag{41}$$

These calculations can be repeated to find y_{eq} moving from Eq. (36).

This time average can be transformed into an ensemble average over the population states. To make this transformation we can imagine that the initial condition is set to (x_{min}, y_{eq}), where x_{min} is the minimum value that the x-population can have during a cycle. The trajectory will then evolve in a counter-clockwise direction, see Fig. 1 for example. It is convenient to divide the integral in Eq. (41) into two contributions

$$x_{eq} = \frac{1}{T}\left(\int_0^{T'} \mathrm{d}t\, x(t) + \int_{T'}^{T} \mathrm{d}t\, x(t)\right). \tag{42}$$

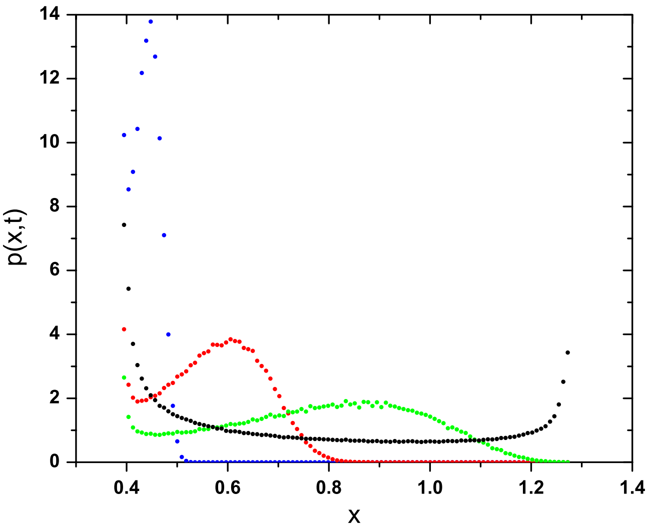

Fig. 3. The probability density for a single subordinated Lotka–Volterra trajectory to have the prey level x at experimental times $t_1 = 10^3$ (blue), $t_2 = 2 \times 10^3$ (red), $t_3 = 3 \times 10^3$ (green), and $t_4 = 10^5$ (black). The average x-value at time t_4 is $x_{avg} = 0.7496$, indicating that $p(x, t_4)$ is approximately the equilibrium distribution, and the average x-value taken from the equilibrium distribution is equivalent to the fixed point $x_{eq} = 0.75$. $N = 10^6$ units were in the ensemble. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Here T' is the time corresponding to the moment when the lower part of the cycle, where $y < y_{eq}$, is completed, that is $x(T') = x_{max}$, the maximum value that the x-population can have. Now we make a change of variables from time to population space, using the x-velocity, $v_x = \frac{\mathrm{d}x}{\mathrm{d}\tau}$ defined in Eq. (36),

$$x_{eq} = \frac{1}{T}\left(\int_{x(0)}^{x(T')} \mathrm{d}x \frac{x}{v_x(x, y^-)} + \int_{x(T')}^{x(T)} \mathrm{d}x \frac{x}{v_x(x, y^+)}\right). \tag{43}$$

The x-velocity is positive during the lower part of the cycle where $y < y_{eq}$. For this reason we label the x-velocity in this region as $v_x(x, y^-)$. On the upper part of the cycle where $y > y_{eq}$, the x-velocity is negative, and we write $v_x(x, y^+)$ to distinguish this case from the lower part of the cycle. Now using the fact that $x(0) = x(T)$ we can combine the integrals on the right-hand side of Eq. (43) to get the result

$$x_{eq} = \int \mathrm{d}x\, x\, p_{eq}(x) \tag{44}$$

where the equilibrium *pdf* depends inversely on the velocity through

$$p_{eq}(x) = \frac{1}{T}\left(\frac{1}{v_x(x, y^-)} - \frac{1}{v_x(x, y^+)}\right). \tag{45}$$

Using this ensemble averaging approach, the equilibrium point reached by the fractional trajectory can be understood intuitively through the equivalent subordination process. Imagine an ensemble of ordinary Lotka–Volterra systems all identically prepared with the initial condition $\mathbf{V}(0) = (x_0, y_0)$ at time $t = 0$. Each system then evolves along the cycle, but making intermittent steps in time according to its own internal clock. The probability density $p(\mathbf{V}, t)$ to be at the coordinate $\mathbf{V} = (x, y)$ at time t spreads from the initial delta function, eventually covering all possible points on the cycle, as some systems step ahead while others get stuck for extended periods. After a sufficiently long time an equilibrium *pdf* $p_{eq}(\mathbf{V})$, that is proportional to the inverse of the x and y velocities, is reached. The *pdf* for a single subordinated trajectory to have a prey level x at four specific experimental times is shown in Fig. 3. The initial delta function spreads to cover the entire range of possible x-values, and tends to the equilibrium *pdf* $p_{eq}(x)$. The first moment of the prey distribution in the asymptotic time limit tends toward the fixed point as predicted by Eq. (44). An equivalent analysis can be repeated for the predator population.

The time required for this equilibrium condition to be reached depends on the index α of the fractional derivative. For $\alpha = 1$ there is no decorrelation between the identical systems, and all travel together along the cycle. Therefore we can intuitively expect that for $\alpha \to 1$ the time to reach equilibrium becomes virtually infinitely extended as the decorrelation among the systems is an extremely slow process.

Moving from a discrete operational time to a continuous time representation [37], Eq. (11) becomes

$$\mathbf{V}(t) = \int_0^\infty \mathrm{d}\tau\, p^{(S)}(t, \tau)\mathbf{V}(\tau) \tag{46}$$

where $p^{(S)}(t, \tau)$ is the probability that the operational time τ is subordinated to the experimental time t, and for a scaling *pdf*

$$p^{(S)}(t, \tau) = \frac{1}{t^\alpha} F\left(\frac{\tau}{t^\alpha}\right). \tag{47}$$

The time derivative of the trajectory is subsequently given by the approximate expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}(t) \approx \frac{\alpha}{t^{1-\alpha}} \int_0^\infty \mathrm{d}y\, yF(y)\mathbf{V}'(y) \tag{48}$$

where we assume $\mathbf{V}'(yt^\alpha)$ can be replaced by $\mathbf{V}'(y)$ for $\alpha \to 1$.

The time derivative of the trajectory tends toward zero as an inverse power-law, but for α close to one it may take a very long time to achieve its objective. Theoretically, a small region in phase space around the equilibrium point is eventually reached by the fractional trajectory for any $\alpha < 1$. But for all practical purposes, when α is close to unity the decorrelation process is so slow that the equilibrium condition would never be realized in numerical calculations.

4. Concluding remarks

We have demonstrated a general connection between fractional calculus and the process of subordination. The significance of this connection is that a single fractional trajectory obtained as the solution to a fractional equation of motion can be interpreted as an average over an ensemble of stochastically evolving ordinary trajectories. This interpretation provides an increased physical understanding of the fractional calculus. Specifically, by adopting a decorrelation perspective rather than the notion of an intrinsic friction, we are able to predict the time it takes for fractional systems to reach equilibrium using subordination theory.

It should be noted that accurate numerical integration of fractional systems is an area of active research. At the present time there is no conclusive evidence that the fractional trajectory resulting from the predictor–corrector integration accurately represents the true theoretical fractional trajectory. We leave it as an open problem to determine the best numerical approach for the study of a given fractional system, which would involve a detailed comparison of several methods of fractional integration, as well as subordination. Regardless of this limitation, the work presented here is intended to assist in improving the accuracy of the current numerical methods. For example, implementing some fractional integration methods may result in a fractional trajectory that at short times lies outside the original Lotka–Volterra cycle. This effect appears to be evidence of poor accuracy. It cannot be realized through a subordination process, where decorrelation always leads to a trajectory lying completely inside the original cycle.

The new insight that the equivalent subordination process brings to the interpretation of fractional trajectories is far-reaching in its implications for complex and chaotic systems. It is well-known that memory effects can suppress chaos [38], and fractional chaotic systems are attracting the attention of many researchers. With an ensemble average perspective the explanation of the behavior of fractional chaotic systems changes dramatically. Chaos has been found to persist in fractional systems when α remains close to unity [39]. On the basis of our results we are led to believe this may be a consequence of a finite observation time. If the observation is extended infinitely, we expect that chaos would eventually be suppressed for any fractional system having α smaller than one. Therefore, adopting the new perspective, the stability analysis of these fractional systems can be significantly improved from the current level of understanding. This is moving in a promising direction as the connection between fractional calculus and the complex processes found in nature continues to grow.

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